Symmetries in Non Commutative Configuration space. *

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Abstract

Extending earlier work [7], we examine the deformation of the canonical symplectic structure in a cotangent bundle $T^\star(\mathcal{Q})$ by additional terms implying the Poisson non-commutativity of both configuration and momentum variables. In this short note, we claim this can be done consistently when $\mathcal{Q}$ is a Lie group.

1 Introduction

When a symplectic manifold is a cotangent bundle $\kappa : T^*(\mathcal{Q}) \to \mathcal{Q}$ with its canonical symplectic structure $\omega_0 = dq^i \wedge dp_i$, the action of a diffeomorphism ϕ on $\mathcal{Q}$ induces a diffeomorphism Φ on $T^*(\mathcal{Q})$ conserving ω_0 :

$$\Phi : T^\star(\mathcal{Q}) \to T^\star(\mathcal{Q}) : \{q^i, p_k\} \to \left\{ {q'}^i = \phi^i(q), p'_k \right\} \; ; \; p_l = p'_k \, \frac{\partial \phi^k(q)}{\partial q^l} \qquad (1)$$

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In particular a group action being a homomorphism $G \rightarrow \mathbf{Diff}(\mathcal{Q})$, induces a strictly Hamiltonian action on $T^*(\mathcal{Q})$:

$$\Phi_g : T^\star(\mathcal{Q}) \rightarrow T^\star(\mathcal{Q}) : (q^i, p_k) \rightarrow \left(q'^i = \phi^i(g,q), p'_k\right) \; ; \; p_l = p'_k \, \frac{\partial \phi^k(g,q)}{\partial q^l} \quad (2)$$

Let $\mathbf{F}$ be a closed two-form on configuration space, then it is well known [1] that a change in the symplectic structure, $\omega_0 \rightarrow \omega_1 = \omega_0 + \kappa^\star \mathbf{F}$, induces a "magnetic" interaction without changing the "free" Hamiltonian. With this new symplectic structure, the momenta variables cease to Poisson commute and one needs to introduce a potential to switch to Darboux variables.
It is then tempting to introduce also a closed two-form in the p-variables in such a way that Poisson non commuting q-variables will emerge[1]. In this way, we obtain a (pre-)symplectic structure :

$$\omega = \omega_0 - \frac{1}{2} \, F_{ij}(q) \, dq^i \wedge dq^j + \frac{1}{2} \, G^{kl}(p) \, dp_k \wedge dp_l \; ; \; d\omega = 0 \qquad (3)$$

Obviously the structure of such a two-form is not maintained by general diffeomorphisms of type **(1)**. But for an affine configuration space, there is the privileged group of affine transformations, $q^i \rightarrow q'^i = A^i{}_j \, q^j + b^i$, which conserve such a structure. When an origin is fixed, this configuration space is identified with the translation group $\mathcal{Q} = G \equiv \mathbf{R}^N$ with commutative Lie algebra $\mathcal{G} \equiv \mathbf{R}^N$ and dual $\mathcal{G}^\star \equiv \mathbf{R}^{*N}$. Furthermore, if $\mathbf{F}$ and $\mathbf{G}$ are constant, ω is invariant under translations. Such a situation was examined for the N-dimensional case in our previous work [7]. From the work of Souriau and others [1, 2, 4, 5] it is clear how to generalize the first term of this extension of the canonical symplectic two-form when configuration space is a Lie group G such that phase space is trivialised $T^*G \approx G \times \mathcal{G}^\star$. This is done introducing a symplectic one-cocycle, defined below.

2 The symplectic one-cocycle

A 1-chain θ on $\mathcal{G}$ with values in $\mathcal{G}^\star$, on which $\mathcal{G}$ acts with the coadjoint representation $\mathbf{k}$, $\theta \in C^1(\mathcal{G}, \mathcal{G}^\star, \mathbf{k})$, is a linear map $\theta : \mathcal{G} \rightarrow \mathcal{G}^\star : \mathbf{u} \rightarrow \theta(\mathbf{u})$.

[1] Such an approach towards non commutative coordinates was originally proposed in [6] in the two-dimensional case with posible application to anyon physics.

Let $\{\mathbf{e}_\alpha\}$ be a basis of the Lie algebra $\mathcal{G}$ with dual basis $\{\epsilon^\beta\}$ of $\mathcal{G}^\star$ and structure constants $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\mu\,{}^\mu\mathbf{f}_{\alpha\beta}$. The 1-cochain is given by $\theta(\mathbf{u}) = \theta_{\alpha,\mu}\, u^\mu\, \epsilon^\alpha$, where $\theta_{\alpha,\mu} \doteq \langle \theta(\mathbf{e}_\mu) | \mathbf{e}_\alpha \rangle$.
It is a 1-cocycle, $\theta \in Z^1(\mathcal{G}, \mathcal{G}^\star, \mathbf{k})$, if it has a vanishing coboundary:

$$
\begin{aligned}
(\delta_1\theta)(\mathbf{u},\mathbf{v}) &\doteq \mathbf{k}(\mathbf{u})\theta(\mathbf{v}) - \mathbf{k}(\mathbf{v})\theta(\mathbf{u}) - \theta([\mathbf{u},\mathbf{v}]) = 0 \\
\langle (\delta_1\theta)(\mathbf{u},\mathbf{v}) | \mathbf{w} \rangle &\doteq -\langle \theta(\mathbf{v}) | [\mathbf{u},\mathbf{w}] \rangle + \langle \theta(\mathbf{u}) | [\mathbf{v},\mathbf{w}] \rangle - \langle \theta([\mathbf{u},\mathbf{v}]) | \mathbf{w} \rangle = 0 \\
(\delta_1\theta)_{\alpha,\mu\nu} &\doteq \langle (\delta_1\theta)(\mathbf{e}_\mu,\mathbf{e}_\nu) | \mathbf{e}_\alpha \rangle \\
&\doteq -\theta_{\kappa,\nu}\,{}^\kappa\mathbf{f}_{\mu\alpha} + \theta_{\kappa,\mu}\,{}^\kappa\mathbf{f}_{\nu\alpha} - \theta_{\kappa,\alpha}\,{}^\kappa\mathbf{f}_{\mu\nu} = 0
\end{aligned}
$$

The 1-cocycle is called symplectic if $\Theta(\mathbf{u},\mathbf{v}) \doteq \langle \theta(\mathbf{u}) | \mathbf{v} \rangle$ is antisymmetric :

$$
\Theta(\mathbf{u},\mathbf{v}) = -\Theta(\mathbf{v},\mathbf{u}) \; ; \; \Theta_{\alpha\mu} \doteq \theta_{\alpha,\mu}
$$

Any antisymmetric Θ defined in terms of $\theta \in C^1(\mathcal{G}, \mathcal{G}^\star, \mathbf{k})$ is actually a 2-cochain on $\mathcal{G}$ with values in $\mathbf{R}$ and trivial representation : $\Theta \in C^2(\mathcal{G}, \mathbf{R}, \mathbf{0})$. Furthermore, when $\theta \in Z^1(\mathcal{G}, \mathcal{G}^\star, \mathbf{k})$, Θ is a 2-cocycle of $Z^2(\mathcal{G}, \mathbf{R}, \mathbf{0})$:

$$
(\delta_2\Theta)(\mathbf{u},\mathbf{v},\mathbf{w}) \doteq -\Theta([\mathbf{u},\mathbf{v}],\mathbf{w}) + \Theta([\mathbf{u},\mathbf{w}],\mathbf{v}) - \Theta([\mathbf{v},\mathbf{w}],\mathbf{u}) = 0
$$

$$
(\delta_2\Theta)(\mathbf{e}_\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma) \doteq -\Theta_{\kappa\gamma}\,{}^\kappa\mathbf{f}_{\alpha\beta} + \Theta_{\kappa\beta}\,{}^\kappa\mathbf{f}_{\alpha\gamma} - \Theta_{\kappa\alpha}\,{}^\kappa\mathbf{f}_{\beta\gamma} = 0 \tag{4}
$$

When $\mathcal{G}$ is semisimple, Θ is exact. Indeed, the Whitehead lemma's state that $H^1(\mathcal{G}, \mathbf{R}, \mathbf{0}) = 0$ and $H^2(\mathcal{G}, \mathbf{R}, \mathbf{0}) = 0$. So, Θ is a coboundary of $B^2(\mathcal{G}, \mathbf{R}, \mathbf{0})$ and there exists an element ξ of $C^1(\mathcal{G}, \mathbf{R}, \mathbf{0}) \equiv \mathcal{G}^\star$ such that $\Theta(\mathbf{u},\mathbf{v}) = (\delta_1(\xi))(\mathbf{u},\mathbf{v}) = -\xi([\mathbf{u},\mathbf{v}])$ or $\Theta_{\alpha\beta} = -\xi_\mu\,{}^\mu\mathbf{f}_{\alpha\beta}$.
In general, $\Theta = \frac{1}{2}\,\Theta_{\alpha\beta}\,\epsilon^\alpha \wedge \epsilon^\beta$, with Θ obeying the cocycle condition **(4)**. Acting with $L^\star{}_{g^{-1}|g} : T^\star_e(G) \to T^\star_g(G)$, yields the left-invariant forms :

$$
\begin{aligned}
\epsilon^\alpha_L(g) &\doteq L^\star{}_{g^{-1}|g}\,\epsilon^\alpha = L^\alpha{}_\beta(g^{-1};g)\,\mathbf{d}g^\beta \\
\Theta_L(g) &\doteq L^\star{}_{g^{-1}|g}\,\Theta = (1/2)\,\Theta_{\alpha\beta}\,\epsilon^\alpha_L(g) \wedge \epsilon^\beta_L(g)
\end{aligned}
$$

where $L^\alpha{}_\beta(g;h) \doteq \partial(gh)^\alpha/\partial h^\beta$. Using the cocycle relation **(4)** and the Maurer-Cartan structure equations,

$$
\mathbf{d}\epsilon^\alpha_L(g) = -\frac{1}{2}\,{}^\alpha\mathbf{f}_{\mu\nu}\,\epsilon^\mu_L(g) \wedge \epsilon^\nu_L(g)
$$

it is seen that $\Theta_L(g)$ is a closed left-invariant two-form on G.

3 G Actions on $T^\star(G)$

Natural coordinates of points $x = (g, \mathbf{p}) \in T^\star(G)$ are given by (g^α, p_β), where $\mathbf{p} = p_\beta \, \mathbf{d} g^\beta$. There are two canonical trivialisations of the cotangent bundle.

- The left trivialisation :

  $$\lambda : T^\star(G) \to G \times \mathcal{G}^\star : (g, p_g) \to \Big(g, \pi^L = L^\star{}_{g|e} \, p_g = \pi^L{}_\mu \, \epsilon^\mu\Big)_{\mathbf{B}}$$

  which yields "body" coordinates, given by $(g^\alpha, \pi^L{}_\mu)_{\mathbf{B}}$.

- The right trivialisation :

  $$\rho : T^\star(G) \to G \times \mathcal{G}^\star : (g, p_g) \to \Big(g, \pi^R = R^\star{}_{g|e} \, p_g = \pi^R{}_\mu \, \epsilon^\mu\Big)_{\mathbf{S}}$$

  which yields "space" coordinates, given by $(g^\alpha, \pi^R{}_\mu)_{\mathbf{B}}$.

They are related by : $\pi^R = R^\star{}_{g^{-1}|g} \circ L^\star{}_{g|e} \, \pi^L = \mathbf{K}(g) \, \pi^L$, where $\mathbf{K}(g)$ is the coadjoint representation of G in $\mathcal{G}^\star$.
Lifting the left multiplication of G by G to the cotangent bundle yields

$$\Phi^L_a : T^\star(G) \to T^\star(G) : x = (g, p_g) \to y = (ag, p'_{ag} = L^\star_{a^{-1}|ag} \, p_g)$$

From $\lambda \circ L^\star_{a^{-1}|ag} : p_g \to L^\star_{ag|e} \circ L^\star_{a^{-1}|ag} \, p_g = L^\star_{g|e} \, p_g = \pi$, it is seen that, in body coordinates, $\left(\Phi^L_a\right)_{\mathbf{B}} \doteq \lambda \circ \Phi^L_a \circ \lambda^{-1} : (g, \pi^L)_{\mathbf{B}} \to (ag, \pi^L)_{\mathbf{B}}$.
The pull-back of the cotangent projection $\kappa : T^\star(G) \to G : x \doteq (g, \mathbf{p}) \to g$, yields differential forms on the cotangent bundle :

$$\begin{aligned} \langle \epsilon^\alpha_L(x)| &= \kappa^\star_x \, \epsilon^\alpha_L(\kappa(x)) \\ \widetilde{\Theta}_L(x) &= \kappa^\star_x \, \Theta_L(\kappa(x)) = -\,\frac{1}{2}\, \Theta_{\alpha\beta} \, \langle \epsilon^\alpha_L(x)| \wedge \langle \epsilon^\beta_L(x)| \end{aligned} \tag{5}$$

Since $\Theta(g)$ is closed on G, its pull-back, $\widetilde{\Theta}_L(x)$, is a closed 2-form on $T^\star(G)$. Furthermore, the left-invariance of $\epsilon^\alpha_L(g) : L^\star_{a^{-1}|ag} \, \epsilon^\alpha(g) = \epsilon^\alpha(ag)$ implies the Φ^L_a-invariance of its pull-back : $(\Phi^L_a)^\star_{|x} \, \langle \epsilon^\alpha_L(\Phi^L_a(x))| = \langle \epsilon^\alpha_L(x)|$ and so is $\widetilde{\Theta}_L(x)$.
A Φ^L_a-invariant basis of one-forms on $T^\star(T^\star(G))$ is

$$\{\langle \epsilon^\alpha_L|; \langle \mathbf{d}\pi^L{}_\mu|\} \tag{6}$$

The right multiplication by a^{-1} induces another *left* action by :

$$\Phi^R_a : T^\star_g(G) \rightarrow T^\star_{ga^{-1}}(G) : (g, p_g) \rightarrow (ga^{-1}, p'_{ga^{-1}} = R^\star_{a|ga^{-1}}\, p_g)\ ,$$

Computing : $L^\star_{ga^{-1}|e} \circ R^\star_{a|ga^{-1}} \circ L^\star_{g|e}\ \pi^L = L^\star_{a^{-1}|e} \circ R^\star_{a|a^{-1}}\ \pi^L$, it follows that, in body coordinates, Φ^R_a acts as : $\Phi^R_a : (g, \pi^L)_{\mathbf{B}} \rightarrow (g' = ga^{-1}, \pi'^L = \mathbf{K}(a)\pi^L)_{\mathbf{B}}$. Under Φ^R_a, the Φ^L_a-invariant basis **(6)** transforms as

$$\begin{aligned} (\Phi^R_a)^\star_{|x}\ \langle \epsilon^\alpha_L(\Phi^R_a(x))| &= \mathbf{Ad}^\alpha{}_\beta(a)\ \langle \epsilon^\beta_L(x)| \\ (\Phi^R_a)^\star_{|x}\ \langle \mathbf{d}\pi'^L{}_\mu| &= \langle \mathbf{d}\pi^L{}_\nu|\ \mathbf{Ad}^\nu{}_\mu(a^{-1}) \end{aligned} \tag{7}$$

4 The modified symplectic structure on $T^\star(G)$

The canonical Liouville one-form on $T^\star(G)$ and its associated symplectic two-form are $\langle\theta_0| = p_\alpha\ \langle dg^\alpha| = \pi_\mu\ \langle \epsilon^\mu_L|$, and

$$\begin{aligned} \omega_0 &= -\,\mathbf{d}\langle\theta_0| = -\pi_\mu\, \mathbf{d}\langle\epsilon^\mu_L| + \langle\epsilon^\mu| \wedge \langle\mathbf{d}\pi_\mu| \\ &= \frac{1}{2}\,\pi_\mu\ {}^\mu\mathbf{f}_{\alpha\beta}\ \langle\epsilon^\alpha| \wedge \langle\epsilon^\beta| + \langle\epsilon^\mu| \wedge \langle\mathbf{d}\pi_\mu| \end{aligned} \tag{8}$$

A modified symplectic two-form is obtained adding the closed two-form **(5)**, constructed from the symplectic cocycle:

$$\omega = \omega_0 + \widetilde{\Theta}_L = \frac{1}{2}\ (\pi_\mu\ {}^\mu\mathbf{f}_{\alpha\beta} + \Theta_{\alpha\beta})\ \langle\epsilon^\alpha| \wedge \langle\epsilon^\beta| + \langle\epsilon^\mu| \wedge \langle\mathbf{d}\pi_\mu| \tag{9}$$

For semisimple $\mathcal{G}$, this reduces to :

$$\omega = \frac{1}{2}\,(\pi_\mu - \xi_\mu)\ {}^\mu\mathbf{f}_{\alpha\beta}\ \langle\epsilon^\alpha| \wedge \langle\epsilon^\beta| + \langle\epsilon^\mu| \wedge \langle\mathbf{d}\pi_\mu| = -\,\mathbf{d}\ ((\pi_\mu - \xi_\mu)\ \langle\epsilon^\mu_L|) \tag{10}$$

This means that the Liouville form is modified $\langle\theta_L| = ((\pi_\mu - \xi_\mu)\ \langle\epsilon^\mu_L|)$ such that $\omega = -\,\mathbf{d}\langle\theta_L\,|$ and that $\{g, p'_g \doteq L^\star{}_{g^{-1}|g}(\pi - \xi)\}$ and there are global Darboux coordinates : $\{g^\alpha, p'{}_\mu = p_\mu - \xi_\beta\, L^\beta{}_\mu(g^{-1}; g)\}$.
Finally we may add another left-invariant and closed two-form in the π variables $\widetilde{\Upsilon}_L = (1/2)\ \Upsilon^{\mu\nu}\ \langle\mathbf{d}\pi_\mu| \wedge \langle\mathbf{d}\pi_\nu|$ such that

$$\omega_L = \omega_0 + \widetilde{\Theta}_L + \widetilde{\Upsilon}_L \tag{11}$$

defines a Φ^L_a-invariant (pre)-symplectic two form on $T^\star(G)$.
Under Φ^R_a, this (pre-)symplectic two-form **(12)** is invariant if a belongs to the intersection of the isotropy groups of $\widetilde{\Theta}_L$ and $\widetilde{\Upsilon}_L$:

$$\Theta_{\alpha\beta}\ \mathbf{Ad}^\alpha{}_\mu(a)\, \mathbf{Ad}^\beta{}_\nu(a) = \Theta_{\mu\nu}\ ;\ \mathbf{Ad}^\alpha{}_\mu(a^{-1})\, \mathbf{Ad}^\beta{}_\nu(a^{-1})\ \Upsilon^{\mu\nu} = \Upsilon^{\alpha\beta} \tag{12}$$

5 Conclusions

The degeneracy of the two-form **(11)** will be examined in further work, as was done in [7] for the abelian group. If ω_L is not degenerate, Poisson Brackets can be defined and, in the degenerate case, the constrained formalism of [3] is applicable. Finally, if the isotropy group of **(12)** is not empty, the remaining Φ_a^R-invariance will provide momentum mappings. Equations of motion of the Euler type will follow from a Hamiltonian of the form

$$H \doteq \frac{1}{2}\,\mathcal{I}^{\mu\nu}\,{\pi^L}_\mu\,{\pi^L}_\nu$$

The momenta mentionned above will be conserved if the isotropy group above also conserves the *inertia tensor* $\mathcal{I}$.

References

[1] J-M. Souriau,
Structure des systèmes dynamiques,Dunod,1970.

[2] R. Abraham and J.E. Marsden,
Foundations of Mechanics,Benjamin,1978

[3] M.J. Gotay, J.M. Nester and G. Hinds,
J.Math.Phys.**19**,2388(1978).

[4] P.Liberman and Ch-M.Marle,
Symplectic Geometry and Analytical Mechanics,
D.Reidel Pub.Comp.,1987

[5] J.A. de Azcárraga and J.M.Izquierdo,
Lie groups, Lie algebras, cohomology and some applications in physics,
Cambridge Univ.Press,1998.

[6] P.A. Horváthy,
Ann.Phys.**299**,128(2002)

[7] F.J.Vanhecke, C.Sigaud and A.R.da Silva,
arXiv:math-phys/0502003(2005) and Braz.J.Phys.**36**,no IB,194(2006)